MATH 22

Lecture B: 9/4/2003

COUNTING

How do I love thee?
Let me count the ways.

—Elizabeth Barrett Browning,
*Sonnets from the Portuguese, X*LIII

I counted two and seventy stenches,
All well-defined, and several stinks.

—Samuel Taylor Coleridge, *Cologne*

Tros Tyriusque mihi nullo discrimine agetur.

—Virgil, *Aeneid* I:574

Administrivia

- Roll call
- Academic Resource Center (Dowling Hall)
- Homework, in folders, is due Tuesday 9/9
- http://denenberg.com/LectureB.pdf
- Questions about Lecture A, first problem set, first project?
- Office hours after class today in room 214, or by appointment anytime. (Don't believe anything about office hours Tuesday.)

Today: Counting, rules of sum & product, examples, factorials, permutations, permutations with repetitions, combinations.

I shall recognize no difference between the Tyrian and the Trojan.

Counting

The area of discrete math called *combinatorics* often asks the question "How many different ways can something happen?"

- How many different poker hands are there?
- How many different ways can two dice land?
- How many different ways are there to pick four baseball teams given 36 people?
- How many different ways are there to fully parenthesize the expression a*b+c*d-e+f*g?
- How many distinct bytes of memory can be addressed by a 32-bit word?

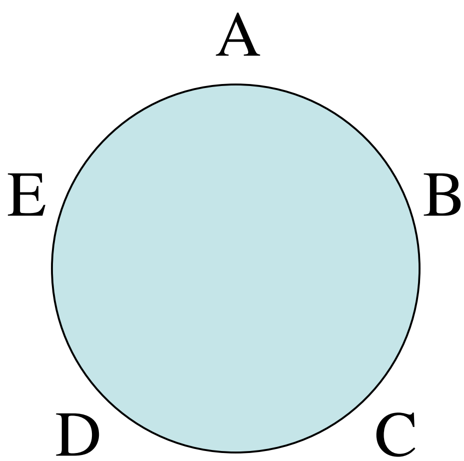
(For example, this question constantly arises in calculating probabilities, as we'll see later.)

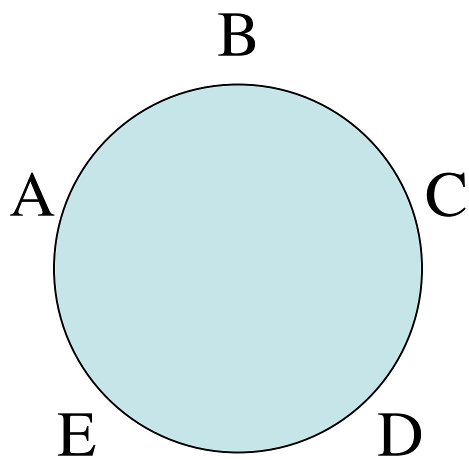
Counting, also known as *enumeration*, can be one of the hardest problems in mathematics. In this course we'll only get the slightest taste of its complexity.

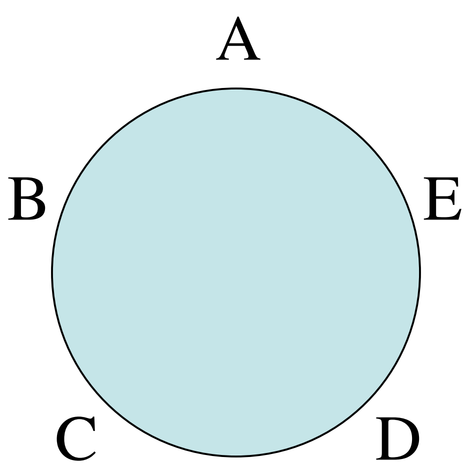
The absolutely crucial word in all of these questions is the word *different* (or *distinct*).

What does *different* mean?

Problem: How many different ways are there to seat five people, A, B, C, D, and E, around a circular table (whose seats don't move)?







Are these three arrangements distinct? Or all the same? Or maybe the upper two are the same but the third is different? What if it were keys on a ring rather than people around a table? The answer to the problem depends on what you want "different" to mean!

(As an example of how tricky this can be, consider Example 1.22 on p. 16, which Grimaldi gets wrong.)

Rule of Sum

Suppose there are m ways to do something, and there are n ways to do something *different*, but you only want to do one thing or the other. Then there are $m+n$ things you can do.

Example: If a store has 12 records by the Beatles, and 15 records by the Rolling Stones, and you can buy only one record, then you have a choice of 27 records.

Frankly, I find this “rule” almost too silly to require explicit formulation. You mostly just apply it instinctively. But consider:

Example: If one store has 12 records by the Beatles, and another store has 15 records by the Beatles, and you can buy only one record, then you may or may not have a choice of 27 different records. It may be as few as 15!

There is a beautiful generalization of this rule called the Principle of Inclusion and Exclusion that covers this and much more complex cases.

Rule of Product

Suppose there are m ways to do a first thing, and n ways to do a second thing. Then there are mn ways to do both things (the first followed by the second, in order).

Example: If a store has 12 records by the Beatles, and 15 records by the Rolling Stones, and you want one of each, then there are (12)(15) ways you can spend your $.

This rule is used in zillions of counting problems. (Again, it's pretty instinctive.) Before giving examples, let's generalize:

Suppose there are m_1 ways to do a first thing, m_2 ways to do a second thing, m_3 ways to do a third thing, . . ., and m_k ways to do a k^{th} thing. Then there are $m_1 m_2 m_3 \ldots m_k$ ways to do the k things successively, one after another.

Example: If a store has 12 records by the Beatles, 15 by the Rolling Stones, and 7 by Rudy Vallee, and you want one of each, then there are (12)(15)(7) ways you can buy three records.

Less Trivial Examples

Example: How many three-letter words are possible?

We must pick the first letter, then the second, then the third. There are 26 choices in each case. So there are 26^3 possible three-letter words. Wow. Qqq. Zxy.

Example: How many four-digit numbers are there?

We must pick the first digit, then the second, the third, the fourth. For the first digit we have nine choices (it can't be 0), for the other three we have ten choices each. So there are (9)(10)(10)(10) = 9000 four-digit numbers.

Example: A bit can be 0 or 1. How many 32-bit words are possible?

We have to pick the first bit (2 choices), then the second (2 choices), etc. for 32 bits. So there are 2^{32} possibilities. (Be sure you understand why it's not 32^2!!)

An Even Less Trivial Example

A Nebraska license plate consists either of 3 letters followed by 3 digits, or 4 letters followed by 2 digits, e.g. ABC838, AAA111, ABCD12, XXXX33 (but not AB1212 or 123ABC). How many Cornhusker cars can there be?

We break this problem into two cases. First consider the 3-letter plates. There are

$$(26)(26)(26)(10)(10)(10)$$

of these. Similarly, there are

$$(26)(26)(26)(26)(10)(10)$$

plates with 4 letters and 2 digits. Since each plate is one or the other *and not both,* the total number of plates is the sum of these two values (which I'm far too lazy to compute).

Note how we've used both the Rule of Sum and the Rule of Product here.

It's critical that our two cases are *separate*: no plate appears in both cases!

A Bogus Example

An Illinois license plate consists of six digits, where *either* each digit is 6 or greater, *or* each digit is odd. For example, 886796 or 987899 or 313793 or 991135, but not 123456. How many Hoosier cars can there be?

Again, break the problem into two cases. First consider the plates with digits 6 or greater; there are four such digits and hence 4^6 plates.

Next consider the plates with odd digits; there are 5 odd digits and hence 5^6 such plates. So far so good.

But it's *wrong* to say that the total number of plates is 4^6+5^6, because the two cases aren't separate as required by the Rule of Sum. License plate 799779 falls into *both* cases (as do many others) and therefore has been counted twice! We say that we've made a mistake of *double counting*. The correct answer is *smaller* than 4^6+5^6. (See Venn.)

Permutations

A very important application of the Rule of Product happens when we're selecting items from a fixed set and can use each only once.

How many three-letter words are possible if no letter may appear more than once?

The answer is not (26)(26)(26) because we can't reuse letters: AAA is not permitted. The answer is that we can pick the first letter in 26 ways, the second letter in 25 ways (since the first letter is no longer valid), and the third letter in 24 ways. The answer is (26)(25)(24).

How many ten-digit numbers are there in which no digit is repeated?

The first digit can be any of 10, the second any of 9, the third any of 8, etc. The answer is

$$(10)(9)(8)(7)\ldots(2)(1).$$

[How many eleven-digit numbers are there in which no digit is repeated?]

Factorials

(This slide is a digression, purely about *notation*.)

We need a handy way to represent products of the form

$$(n)(n-1)(n-2)\ldots(3)(2)\ (1)$$

We write $n!$ and say "*n* factorial" (or "*n* bang") to denote this product. Examples:

$$5! = (5)(4)(3)(2)(1) = 120$$

$$2! = (2)(1) = 2$$

$$1! = 1$$

Useful facts:

$$n! \,/\, k! \;=\; (n)(n-1)(n-2)\ldots(k+1) \qquad \text{[why?]}$$

$$n! \;=\; n(n-1)! \qquad \text{[why?]}$$

This last fact helps us to settle a puzzling case: What is zero factorial (an important quantity)? We must have

$$1! \;= 1(0!)$$

which makes it clear that $0!$ must equal 1. (This is also true because a product of zero factors must equal 1, just as a sum of zero terms must equal 0. Don't worry if you don't understand that last sentence.)

And what about $-1!$? Well, we have $0! \;=\; 0(-1!)$ which means that $-1! \;= 1/0$, that is, $-1!$ doesn't exist. And neither do $-2!$, $-3!$, $-4!$, etc. But $1.5!$ *does* exist (as does $-1.5!$), though that's beyond the scope of this course.

Back to Permutations

In general, suppose we have n different items. How many ways there are to arrange r of these items in order without reusing any? The answer is called the *number of permutations of n things taken r at a time,* written $P(n,r)$ in this class but ${}_nP_r$ in days of yore.

The answer is that the first item can be any of the n, the second item can be any of the n-1 remaining, the third item can be any of the n-2 remaining, etc., and the r^{th} item can be any of the (n-r+1) remaining. So the answer is

$$P(n,r) = (n)(n-1)(n-2)\ldots(n-r+1) = n! / (n-r)!$$

In the three-letter word problem we have n=26 and r=3, so the answer is $P(26,3) = 26! / 23!$. (This is the answer, by the way. Not 15600.)

In the ten-digit number problem we have n=r=10, giving $P(10,10) = 10! / 0! = 10!$. It's easy to see that $P(n,n) = n!$ for all n.

Permutations with Repetitions

A twist: What happens when the objects to be permuted aren't different?

How many permutations are there of (all) the four letters A, B, C, D?

Answer: P(4,4) = 4! = 24.

But how many permutations are there of the four letters A, A, A, A?

Answer: Only one, namely AAAA. We can choose a first A then a second, etc., but they're all the same A!

How many permutations are there of the four letters A, A, B, B? Or of the letters in the word MISSISSIPPI (the classic example)?

How do we handle this problem in general?

Perms w/ Reps, cont.

Here's what we do. Suppose we want to find all the permutations of the letters A, A, B, C. We start by relabelling the two As so that they *are* distinct. The letters are now A_1, A_2, B, C.

We know there are P(4,4) = 4! permutations of these four "letters". So far so good.

Key point: These permutations come in *pairs* whose only difference is the labelling on the two As. For example, A_1BCA_2 and A_2BCA_1 form a pair, as do BA_1A_2C and BA_2A_1C. When we erase the labels on the As, the two elements of the pairs look exactly alike! So the real question we need to answer is: How many pairs are there?

Well, each permutation belongs to exactly one pair, and the pairs are all distinct, so there are 4!/2 pairs. And this is the answer.

Forging ahead

Next step: How many permutations are there of the letters A,A,A,B,C? Again, label the As so they're distinct, giving A_1, A_2, A_3, B, C. We know there are $P(5,5)=5!$ permutations of these five distinct letters.

Key point: These permutations come in clumps of 6 whose only difference is the labelling on the As. One clump, for example, is $A_1BA_2A_3C$, $A_1BA_3A_2C$, $A_2BA_1A_3C$, $A_2BA_3A_1C$, $A_3BA_1A_2C$, $A_3BA_2A_1C$. Again, everything in a clump is the same if you erase the labels, so all we need is the number of clumps, which is $5! / 6$.

[Where did "6" come from in this example? It's the number of ways of permuting the three As; each clump contains every possible way of permuting A_1, A_2, A_3, and there are $P(3,3)=3!=6$ such ways.]

Forg$_1$ing$_2$ a$_1$hea$_2$d

What if more than one letter is duplicated? How many different permutations are there of the letters A,A,A,B,B,B,B,C? The answer is that the clumps now have size 3!4! (there are 3! ways of permuting the As and 4! ways of permuting the Bs, and a member of a clump has one of each). If we keep the As and Bs distinct there are 8! permutations, so there must be 8!/4!3! clumps. And as before, the number of clumps is the answer.

The book gives the general formula. You should now be able to tell how many ways there are of rearranging the letters of MISSISSIPPI: it's 11! / (4!)(4!)(2!).

[Comment: In all the perms-with-dups examples we've asked for the number of reorderings of *all* the letters. But how many ways are there to make a five-letter word from the letters in MISSISSIPPI? If you try to work this out you'll come across a problem: The clumps don't have the same size, so you can't just divide! Solving this seemingly-only-slightly-more-difficult problem is beyond the scope of this course.]

Ignoring Order

All the work we've done so far has been with *arrangements*, that is, *ordered* selections. (For example, CAT and ACT are different three-letter words. We choose the first letter, then the second, then the third. If we weren't doing it this way, we couldn't apply the Rule of Product.)

But now: What if order doesn't matter?

How many different ways are there of picking a committee of three from a group of 26 people?

Suppose we label the people A, B, C, . . ., Z. The answer is *not* $P(26,3) = (26)(25)(24)$ because the order isn't important; a committee with C, A, and T is the same as a committee with A, C, and T. The order is irrelevant, and the Rule of Product doesn't apply directly.

What to do?

Clumps Again

Clumps to the rescue! Suppose we consider all the *permutations* of the 26 letters taken 3 at a time. These fall into clumps where the letters are the same but the order is different. (For example, one clump consists of ACT, ATC, CAT, CTA, TCA, and TAC.)

Luckily, each clump has the same size; in this example, each clump has size 6. (Why 6? Because there are 6 ways of permuting the three letters in each clump.)

As before, the answer we want is just the number of clumps; in this case (26)(25)(24)/6.

In general, if we want the number of ways of choosing r items out of a set of n distinct items, we consider all $P(n,r)$ *ordered* arrangements of the n items taken r at a time, and we note that they fall into clumps of size $P(r,r) = r!$. So the answer is $P(n,r)/P(r,r) = P(n,r)/r!$.

Combinations

This number, the number of ways of selecting r objects from n distinct objects *where order doesn't matter*, is called the *number of combinations of n things taken r at a time*. It can be written $C(n,r)$ but is most commonly written in a way I can't manage here. It's quite typically pronounced "*n* choose *r*".

To recap, we have

$$C(n,r) = P(n,r) / P(r,r) = n! / (n-r)!r!$$

Example: In the original Mass Megabucks lottery, a ticket was 6 of the numbers 1,2,3,…,36, with order unimportant. How many tickets are there?

Answer: $C(36,6) = 36! / 6!\,30!$

We're going to see a lot of these numbers; they're more important than permutations. Many more examples next time.