MATH 22

Lecture H: 9/25/2003

PRACTICE
& REVIEW

Multiplication is vexation,
Division is as bad;
The Rule of Three doth puzzle me,
And Practice drives me mad.

—Anonymous; in
Lean's Collecteana vol 4. p 53

Administrivia

- http://denenberg.com/LectureH.pdf,
  for what it's worth

- Exam #1
  - Monday, 9/29, 11:50–1:20
  - **Location Change: Pearson Chemistry 106**
  - Covers Chapters 1, 2, and Section 5.5 (basically half counting and half logic)

(As promised, I'm writing out the solutions to the review problems. They're at the end of this document. No complaints about the lack of formatting, please.)

Answers to Emailed Queries

What You Need, Part 1

COUNTING

- How to identify different kinds of situations
  - Permutations vs combinations
  - Whether duplications are permitted
  - Special case “at least one of each kind”
  - Catalan numbers

- A handful of formulas
  - $P(n,r)$, $C(n,r)$, $C(n+r-1, r)$, n^r
  - $(1/(n+1))\,C(2n,n)$ (the Catalan formula)
  - Rule of sum & product
  - The binomial theorem
  - Simple identities (maybe)

- When to use which formula, and how to identify n and r! (The hard part.)

What You Need, Part 2

LOGIC

- Propositional calculus
  - Connectives and their meanings
  - Truth tables
  - Equivalences ("Laws")
  - Simplification

- Predicate calculus
  - Quantifiers and their meaning
  - Quantifier laws and equivalences

- Proofs
  - Proof by truth tables
  - Rules of Inference and their use
  - Proof by contrapositive and contradiction
  - Proofs of real mathematical statements

What You Need, Part 3

THE PIGEONHOLE PRINCIPLE

- If m pigeons occupy n pigeonholes and $m > n$, then at least one pigeonhole contains two or more pigeons.

- If m pigeons occupy n pigeonholes and $m>kn$, then at least one pigeonhole contains $k+1$ or more pigeons.

- If there's anything more to say about this one, please tell me what it is.

Practice Problems

Prove that $P(n,n) = P(n,n-1)$ and explain why this is so using a counting argument.

You have $2n$ objects, but n are identical and the rest distinct (so that there are $n+1$ different kinds of objects). In how many different ways can we select n objects out of these $2n$ objects?

How many divisors does the number 1400 have?

Suppose you have a congress with 3 parties and $2n+1$ indistinguishable seats. How many ways are there to divide the seats among the parties? How many ways are there to divide the seats so that no party has a majority?

In how many ways can the letters a, a, a, a, a, b, c, d, e be permuted such that no two a's are adjacent?

(All these problems were taken without permission from Liu, *Introduction to Combinatorial Mathematics*, my copy of which is signed by the author!)

Practice Problems

Write out a truth table for this formula:

$(((p \quad q)(p \quad \neg q) \quad (\neg p)q) \equiv q) \quad □ \quad (pr \quad p(\neg r))$

Alternate: Simplify the formula as much as possible, giving the rules used in each step.

Find some big hairy proof somewhere in Chapter 2 of Grimaldi and cover up all the reasons. Supply them. Don't forget to give the name of each and the previous steps it applies to. Write out the conditional whose truth table you would have to evaluate in order to do the same proof; note that you would have no way to do this if there are any quantifiers in the proof!

Prove that the square root of 2 is irrational, that is, prove that there do not exist two integers a and b such that $a / b = \sqrt{2}$. Hint: Use contradiction.

We have said that the formulas $(□x)(□y)\ p(x,y)$ and $(□y)(□x)\ p(x,y)$ are *not* equivalent. Recall that an equivalence is the conjunction of two conditionals.

For each conditional, either argue informally why it's true or give an example (that is, a p) that shows it's false.

Practice Problems

[In each case, identify which are the pigeons and which are the holes!]

How many people do you have to gather together in order to guarantee that there will be two who have the same first and last initials?

Show that if 5 points are selected in the interior of a square of side 1, then there must be at least two that are no farther apart then $1/\sqrt{2}$.

You have a zillion socks in your drawer. Some are red, some are blue, some are green, some are purple, some are black, and some are stroup. If you're in the dark, how many socks do you have to pull out to guarantee you have a pair? How many socks do you have to pull out to guarantee that you have n pairs all of the same color?

You are sitting around cleaning your pistol when a flock of m pigeons flies by. Assuming that you never miss, how many bullets do you have to shoot in order to guarantee that there is at least one pigeon with k holes?

Solution

Prove that $P(n,n) = P(n,n-1)$ and explain why this is so using a counting argument.

Since $P(n,r) = n! / (n-r)!$ we have

$$P(n,n) = n! / (n-n)! = n! / 0! = n! / 1 = n!$$

and

$$P(n,n-1) = n! / (n-(n-1))! = n! / 1! = n! / 1 = n!$$

Suppose I write down, in order, all but one of n objects. In how many ways can I append the last object? Only one—there's only one object left, hence only one choice.

Solution

You have $2n$ objects, but n are identical and the rest distinct (so that there are n+1 different kinds of objects). In how many different ways can we select n objects out of these $2n$ objects?

How many selections are there that contain 0 of the identical objects? We must select n of the n distinct objects, and this can be done in $C(n,n)$ ways.

How many selections are there that contain 1 of the identical objects? We select any identical object (they're all the same, so there's only one way to do this) and then we must select n–1 of the others, which can be done in $C(n,n-1)$ ways.

Similarly, how many selections are there that contain i of the identical objects? There's only one way to pick the i, and then there are $C(n, n-i)$ ways to pick the others.

All of these groups of selections are distinct, so we can add them up. The total is

$$C(n,n) + C(n,n-1) + C(n,n-2) + \ldots + C(n, 0)$$

which, as we once proved, is 2^n.

Solution

How many divisors does the number 1400 have?

First note that $1400 = (2^3)(5^2)(7)$. Any divisor of 1400 must be a product of some of these prime factors, and any product of these prime factors will divide 1400.

How many ways can we make a product out of these factors? We can multiply together

0, 1, 2, or 3 factors of 2

0, 1, or 2 factors of 5

0 or 1 factors of 7

That is, there are 4 ways to choose the number of 2s to multiply, 3 ways to choose the number of 5s, and two ways to choose the number of 7s. By Rule of Product, there are $(4)(3)(2) = 24$ divisors of 1400.

Note that this result includes 1 (where we choose to multiply in zero 2s, zero 5s, and zero 7s) and also 1400 (where we multiply three 2s, two 5s, and one 7).

Solution

Suppose you have a congress with 3 parties and $2n+1$ indistinguishable seats. How many ways are there to divide the seats among the parties? How many ways are there to divide the seats so that no party has a majority?

If we just want to divide the seats among the parties, we're asking for the number of positive integer solutions to the equation

$$x_1 + x_2 + x_3 = 2n+1$$

where the value of x_i is the number of seats that the i^{th} party gets. And this, in turn, is the number of ways of choosing $2n+1$ donuts from a bakery that has 3 kinds, that is, the number of ways of choosing $r=2n+1$ things from a set of $n=3$ with duplications allowed; the answer is $C(2n+3, 2n+1)$.

Now we want to subtract the number of ways in which some party has a majority. In how many ways can party 1 have a majority? We count them by giving $n+1$ seats to party 1, then distributing the remaining n seats among the 3 parties, which can be done in $C(n+2, n)$ ways. Similarly, each of the other parties can have a majority in $C(n+2,n)$ ways. Subtracting these, the answer to the original problem is $C(2n+3, 2n+1) - 3\,C(n+2,n)$, or more simply $C(2n+3, 2) - 3\,C(n+2, 2)$.

Solution

In how many ways can the letters a, a, a, a, a, b, c, d, e be permuted such that no two a's are adjacent?

This is kinda easy. The only way to separate the a's is to have them be at the first, third, fifth, seventh, and ninth positions, with the four other letters between. So the answer is the number of ways to order four letters taken out of four, that is, P(4,4), which is 4!

What if the problem asked the same thing about the letters a, a, a, b, c, d, e ?

Solution

Write out a truth table for this formula:

(((*p*   *q*)(*p*   ¬*q*)   (¬*p*)*q*) ≡ *q*) □   (*pr*   *p*(¬*r*))

Alternate: Simplify the formula as much as possible, giving the rules used in each step.

The truth table would have 8 rows and I'm not writing it out. But let's simplify the formula. Note that in simplifying we're allowed to use only Laws of Logic (which are equivalences), not the Laws of Inference (which are only implications!).

The right hand side can be changed to p(r   ¬r) by distributivity. But (r   ¬r) is always true [Inverse] so this simplifies just to p by Identity.

(p   q)(p   ¬q) is equivalent to p (q)(¬q) [distributivity]. But (q)(¬q) is always false [Inverse] so this simplifies just to p [Identity]. Now we have p (¬p)q which is equivalent to (p   q) [prove this!]. So the LHS is down to (p q) ≡ q. This is equivalent to ((p q) □ q)(q □ (p q)). The right side of this is always true and can be dropped by Identity.

We're now down to (p q) □ q □ p. Recall that we can write s□ t as (¬s) t, so this is (¬(p q)   q) □ p, which by the same trick is ¬(¬(p q)   q)   p. Using DeMorgan and Double Negation, this is (p q)(¬q)   p which is p(¬q)   q(¬q)   p by distributing. But q(¬q) is always false and can be dropped, leaving p(¬q)   p. And we said this is equivalent to p.

So the whole original formula is just equivalent to p! It's T if p is T and F if p is F.

Solution

Find some big hairy proof somewhere in Chapter 2 of Grimaldi and cover up all the reasons. Supply them. Don't forget to give the name of each and the previous steps it applies to. Write out the conditional whose truth table you would have to evaluate in order to do the same proof; note that you would have no way to do this if there are any quantifiers in the proof!

No chance, pal.

Solution

Prove that the square root of 2 is irrational, that is, prove that there do not exist two integers *a* and *b* such that $a / b = \sqrt{2}$. Hint: Use contradiction.

We will be using the following Lemma, proved in Project 3: If x is an integer, *x* is even if and only if x^2 is even.

We proceed by contradiction. So assume that the statement is false, that is, assume that there *do* exist integers *a* and *b* such that $a/b = \sqrt{2}$. Reduce the fraction a/b to lowest terms, getting a (possibly) new fraction c/d where $c/d = \sqrt{2}$ and *c* has no factors in common with *d*.

Squaring both sides, we see that $c^2/d^2 = 2$, so $c^2 = 2d^2$. Thus c^2 is even (since it's twice an integer). By the Lemma, *c* is even. So $c = 2z$ for some *z* (by the definition of "even"). Substituting, we have $(2z)^2 = 2d^2$, or $4z^2 = 2d^2$, or $2z^2 = d^2$. So d^2 is even. Using the Lemma again, *d* is even. But we've now proved that *c* and *d* are both even; this is impossible, since c/d was in lowest terms! This contradiction completes the proof.

Solution

We have said that the formulas $(\exists x)(\forall y)\ p(x,y)$ and $(\forall y)(\exists x)\ p(x,y)$ are *not* equivalent. Recall that an equivalence is the conjunction of two conditionals.

For each conditional, either argue informally why it's true or give an example (that is, a p) that shows it's false.

Is it true that $(\exists x)(\forall y)\ p(x,y)$ implies $(\forall y)(\exists x)\ p(x,y)$?

Yes, it does. Suppose the LHS is true. This means that $(\forall y)\ p(c,y)$ is true for some specific *c* (by Existential Specification). Now if p(*c*,*y*) is true for all *y*, then certainly for all *y* there is some *x* such that p(*x*,*y*); no matter what *y* you use, *c* works!

Now, does $(\forall y)(\exists x)\ p(x,y)$ implies $(\exists x)(\forall y)\ p(x,y)$?

No, it doesn't. We did this example in class; suppose for example that p(*x*,*y*) means $x+y = 10$. Then surely the LHS is true: for all y there is such an *x* (you can always pick $y = 10 – x$). But the RHS is false: There is no *x* such that $x+y=10$ for all *y*.

Solution

How many people do you have to gather together in order to guarantee that there will be two who have the same first and last initials?

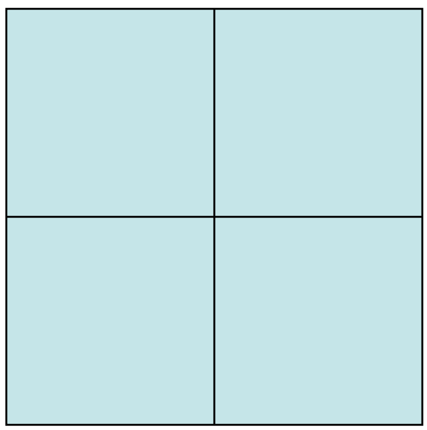
There are $676 = 26^2$ ways of picking two initials. So if we have 677 people, there must be two with the same initials.

The people are the pigeons and the pairs of initials are the holes.

Solution

Show that if 5 points are selected in the interior of a square of side 1, then there must be at least two that are no farther apart then $1/\sqrt{2}$.

Divide the square into four:



The diagonal of each small square is $1/\sqrt{2}$ long, so within a small square two points can't be farther apart than this. But one of the small squares must contain two points by the Pigeonhole Principle!

The points are the pigeons and the small squares are the holes.

Solution

You have a zillion socks in your drawer. Some are red, some are blue, some are green, some are purple, some are black, and some are stroup. If you're in the dark, how many socks do you have to pull out to guarantee you have a pair? How many socks do you have to pull out to guarantee that you have n pairs all of the same color?

The holes are the colors (types of socks) and the pigeons are the socks.

If n=1 you obviously need 7 socks. For general n, remember that n pairs means $2n$ socks! So we need at least $2n$ pigeons in some hole, meaning $6(2n-1)+1$ socks minimum, or $12n-5$ socks. (We got this wrong in class, failing to appreciate the +1/–1 problem. See the next page for another illustration.)

As someone cleverly pointed out, if you want 2 pairs of socks that *needn't* be of the same color, you need to pull out 9 socks. What if you want n pairs that needn't be the same?

Solution

You are sitting around cleaning your pistol when a flock of *m* pigeons flies by. Assuming that you never miss, how many bullets do you have to shoot in order to guarantee that there is at least one pigeon with *k* holes?

The answer is $(k-1)m + 1$.

[Don't get confused by the +1 and –1 here: The generalized PP says that if there are more than km pigeons (i.e., at least km+1) then there is at least one hole with k+1 pigeons. If we need only k pigeons in some hole, as here, we need more than (k–1)m pigeons.]

The holes are the pigeons, and the pigeons are the holes!