MATH 22

Lecture L: 10/9/2003

MORE M.I.; RELATIONS

You know how it is yourself about admiring your relations.

—Cedric Errol, Lord Fauntleroy

Administrivia

- http://denenberg.com/LectureL.pdf
- Yet more on Exam Problem 3 (sigh)

Today: More MI, relations, a start at functions (with luck)

Math. Induction Review

- Well-ordered sets, which we used only to prove . . .

- Principle of Math. Induction: Prove it works for 1; and prove that if it works for n, it also works for n+1

- Examples:

  $1 + 2 + 3 + \ldots + n = n(n+1) / 2$

  $n^3 + 2n$ is divisible by 3

  if $|S| = n$, then $|P(S)| = 2^n$

- Variation: Prove it works for n_0; . . . (needn't start at 1)

  $n^{100} < 2^n$ for all $n \geq 1024$

- M.I., Strong Form: . . . and prove that if it works for everything from 1 to n, it also works for n+1

  $b_0 = b_1 = 1$, $b_n = 2b_{n-1} + b_{n-2}$, then $b_n < 6b_{n-2}$

  $N-1$ links are required to connect N computers

- More examples, to be worked in class

More Examples

Theorem: For every integer $n \geq 2$,

$$1/\ 1 + 1/\ 2 + 1/\ 3 + \ldots + 1/\ n \quad > \quad n$$

Base case: $1/\ 1 + 1/\ 2 \approx 1.7 > 1.414 = \quad n$

Inductive case: We just need to show that

$$1/\ (n+1) \;>\; (n+1) - \ n$$

[Do we believe this? How can we show it?]

Theorem: For every integer $n \geq 0$

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + 3 + \ldots + n)^2$$

Base case: $0 = 0$.

Inductive case: We need to evaluate

$$(1 + 2 + \ldots + n)^2 = ((1 + 2 + \ldots + n) + (n+1))^2$$

Mr. Smith claims to be 1/3 Native American. If asked how this can be, he says “my mother was 1/3 Native American, and my father was 1/3 Native American!” Is this a valid proof by Mathematical Induction?

Convivial Couples

(From Liu) A husband H and wife W invite n couples to dinner. As people arrive, some shake hands. Nobody shakes hands with his or her own spouse. After the handshaking, H asks everyone (including W) how many hands they shook, and no two replies are the same! Prove that W shook hands with exactly n people.

Lemma: With n couples there are $2n+2$ people, and the $2n+1$ replies received by H are 0, 1, 2, 3, . . ., $2n$.

Lemma: If $n > 0$, the person who replied "$2n$" is married to the person who replied "0", and neither one is H or W.

Proof by Mathematical Induction:

(Base case) If $n = 0$, no handshaking happened, so clearly W shook 0 hands.

(Inductive case) If there are $n+1$ couples the second Lemma applies. Eliminate the couple that replied "$2n+2$" and "0". We now are in the same situation with n couples (this requires proof) so, by the inductive hypothesis, W shook n hands. Putting back the last couple, W shook $n+1$ hands. QED

Functions (informally)

(As with relations, we'll do functions informally for awhile.) A *function* is a rule that, given a value, produces another value. Examples:

The "+1" function. Given 3, it produces 4. Given 8, it produces 9. Given x, it produces x+1. Given x^2+7, it produces $x^2 + 8$.

The "squaring" function. Given 5, it produces 25. Given –1, it produces 1. Given x, it produces x^2.

The "father of" function. Given Cain, it produces Adam. Given Larry, it produces Norman.

The "state-located-in" function. Given Natick, it produces MA. Given Council Bluffs, it produces IA.

We write $f(x) = y$ to mean that function f, given x, produces y. So $f_1(4) = 16$ if f_1 is the squaring function, and $f_2(LA) = CA$ if f_2 is the "state-located-in" function.

Note that for the moment *all our functions take a single argument*. We'll worry more about this later.

Useful Functions

Here are some important numeric functions.

For any number *x*, floor(*x*) is the *largest integer less than or equal to x*. This function is also called the "greatest integer" function. We usually write $\lfloor x \rfloor$ for floor(*x*). Examples:

$\lfloor 1.4 \rfloor = 1$ $\lfloor \pi \rfloor = 3$ $\lfloor 3 \rfloor = 3$ $\lfloor -1.5 \rfloor = -2$

(Note the last one carefully.)

For any number *x*, ceiling(*x*) is the *largest integer less than or equal to x*. ceiling(*x*) is written $\lceil x \rceil$. Examples:

$\lceil 1.4 \rceil = 2$ $\lceil 2 \rceil = 2$ $\lceil -3.7 \rceil = -3$

(Again, beware of the negative numbers.)

Much less important is the *truncation* function, which takes any integer and chops off the fractional part:

trunc(1) = trunc(1.87) = 1 trunc(–3.2) = –3

Notice that trunc(*x*) = $\lfloor x \rfloor$ for all nonnegative *x*.

Quickies: What is $\lceil \lfloor x \rfloor \rceil$? What of $-\lfloor -x \rfloor$?

Bunch o' Terms

There's lots of terminology for functions:

A function has to be given a value from a specified set. (You can't give a city to the "squaring" function, nor a number to "father of"!) The set of objects that a function will accept is called the *domain* of the function.

The set of objects that a function might produce is called the *codomain* of the function.

If f is a function with domain *A* and codomain *B*, we say that f is a function from A to B and write $f : A \to B$.

Examples:

The "+1" function has domain and codomain Z (say)

The "father-of" function has domain and codomain equal to the set of humans

The domain of the "state-located-in" function is the set of cities, and its codomain is the set of states

More Terms

If f(x) = y, we sometimes call y the *image of x under f* and we call x a *preimage of y under f*. [Why is y *the* image of x while x is *a* preimage of y?]

Let A be a subset of the domain of f. Then we can write f(A) to denote the set of all values produced by f from "inputs" in A. That is, $f(A) = \{ b \mid b = f(a) \wedge a \in A \}$. For example:

If f is the squaring function, then f([–2,4]) = [0,16].
floor([0.5, 2.9]) = { 0, 1, 2 }

We also call f(A) the *image of A under f*. Note that f(A) is always a set.

The *range* of a function is the set of all values produced by the function. This is not necessarily the same as the codomain. For example, suppose f: $\mathbf{Z} \rightarrow \mathbf{Z}$ is the squaring function. Then the range of f is the *nonnegative* integers, even though the codomain is *all* integers. Note that the range of f is the same as the image of the domain. [time for a blackboard picture]

1-1 Functions

Suppose f is a function, and suppose that f never produces the same result for two different arguments. Then we say that f is a *one-to-one*, or *injective*, function. The formal definition is that f is injective if $f(x)=f(y)$ implies $x=y$. (Do we believe that this is the same thing?)

What does this mean? It means that you can look at the "output" and determine the "input". Examples:

The "+1" function is injective. If $f(x) = f(y)$, that is, if $x+1 = y+1$, then it must be that $x=y$. You can't find two different values that +1 takes into the same value.

The "squaring" function is *not* injective. -3^2 and 3^2 have the same value.

The "father of" function and the "state-located-in" function are not injective. But the "capital-of" function, that takes a state and outputs a city, is injective.

Are floor, ceiling, and trunc injective?

[Blackboard picture here]

Onto Functions

Suppose that f is a function, and for every element *y* of the codomain of f there is some *x* in the domain of f such that f(*x*) = *y*. Then f is an *onto*, or *surjective*, function.

What does this mean? It means that *nothing in the codomain is left out*, everything in the codomain is "hit" by the function. Another way to say this is that a function is surjective if and only if its codomain equals its range.

[Blackboard picture]

Examples:

The +1 function on the integers is surjective, but on the positive integers it's not surjective. The squaring function on the reals is certainly not surjective. The "state-located-in" function is surjective, but not the "capital of" function. What of floor, ceiling, trunc?

(Note that surjectiveness depends critically on what we consider the codomain; by fiddling with the codomain we can change a functions surjectivity while keeping essentially the same function. Not so with injectivity.)

A function that is both injective and surjective is called *bijective*, or sometimes *one-to-one onto*. More on this later.