MATH 22

Lecture S: 11/4/2003

PARTIAL ORDERS

Erst der Krieg schafft Ordnung.
—Brecht, *Mutter Courage*, scene 1

Stand not upon the order of your going.
—Shakespeare, *Macbeth*, III.4

Administrivia

- http://denenberg.com/LectureS.pdf

- Reception in Clarkson Room, Thursday 11/6, 4:30–5:30, to discuss next semester's courses
  (Everyone here will, of course, be late.)

- Comment: Be prepared to take notes today and Thursday. There will be lots of stuff on the blackboard that I can't get into PowerPoint!

This week: Two especially important kinds of binary relations: partial orders and equivalence relations

Only war creates order.

—Bertolt Brecht

Review of Relations

A *binary relation* on a nonempty set S is a subset of $S \times S$, that is, a binary relation is a set of ordered pairs. All of the following expressions mean the same thing:

x bears relationship R to y

x and y are in the R relationship (in that order!)

$(x,y) \in$ R, usually written x R y

Examples of relations

$<$ = { (4,99), (–1,0), (3,4), (3,22), (3, 3233), . . . }

"is within 100 miles of" = { (Boston, Medford), . . . }

"is a sibling of" = { (Cain, Abel), (Larry, Larry), . . . }

A binary relation R on S is

- *reflexive*, if x R x for all $x \in S$
- *symmetric*, if x R y implies y R x for all $x,y \in S$
- *transitive*, if x R y and y R z together imply x R z, for all $x,y,z \in S$
- *antisymmetric*, if x R y and y R x together imply $x = y$, for all $x,y \in S$ [this one is new]
- *irreflexive*, if x R x for *no* $x \in S$ [also new]

Examples

Relation "is within 100 miles of" is reflexive and symmetric, but neither antisymmetric nor transitive.

Relation "is a sibling of" is reflexive [by my definition], symmetric, and transitive, but not antisymmetric.

Relation "is a sister of" is transitive but not reflexive, symmetric, nor antisymmetric.

Relation "loves" has none of the five properties.

Relation $\leq$ is reflexive, transitive, and antisymmetric, but not symmetric. The same is true of relation $\subseteq$.

Relation $<$ is neither reflexive nor symmetric, but is transitive and also irreflexive. [Is it antisymmetric?]

Relation $|$ (divides) is reflexive, transitive, and (on positive integers) antisymmetric. It's not symmetric.

Relation "equals modulo n" is reflexive, transitive, and symmetric, but not antisymmetric.

The empty set, being a subset of $S \times S$, is a relation! It isn't reflexive, but does have the other four properties.

Things to Note

• We're considering only binary relations. But there are also *ternary relations* ("*x* sold *y* to *z*") which are subsets of $S \times S \times S$, *unary relations*, *n-ary relations*, etc.

• All the example relations are "natural" in some way, but a relation can be an *arbitrary* subset of $S \times S$.

• Antisymmetric doesn't mean "not symmetric". A relation can be both (e.g., the relation "=") or neither (e.g., "is a sister of"). Similarly, a relation can be neither reflexive nor irreflexive, though it can't be both.

• Theorem: Suppose relation R on S is symmetric and transitive. Then it must be reflexive.

Proof: Let x be any element of S. Since R is symmetric, $x \text{ R } y$ implies $y \text{ R } x$. But by transitivity, $x \text{ R } y$ and $y \text{ R } x$ together imply $x \text{ R } x$. So we've shown $x \text{ R } x$ for any x in S, which means that R is reflexive.

This theorem is BOGUS! What's wrong with the proof? If R is symmetric and transitive but not reflexive; what property must x have if $x \text{ R } x$ is false?

Partial Orders

Definition: A *partial order* on a set *S* is a binary relation on *S* that is reflexive, transitive, and antisymmetric.

Example: $\leq$, □, and $|$ are partial orders.

To get the intuition behind partial order, let's start with a stronger concept: A *total order* on *S* is a partial order R that satisfies the following property, called *trichotomy*: If x and y are elements of S, then either x R y or y R x (or both, in which case $x = y$ by antisymmetry).

A total order (or total ordering) is a relation that lets us arrange the elements of *S* in order, as though on a line. The most common total order that we know is $\leq$, which arranges numbers in order. Note that $<$ isn't a total order because it's not reflexive.

In a total order, any two elements are related by the order one way or the other (or both). A partial order lacks this property; it's got the other properties of an ordering but may have elements that are *incomparable*, i.e., aren't ordered one way or the other by the relation.

Canonical Example

The first and best example of a partial order is the binary relation "is a subset of", written $\subseteq$: $A \subseteq B$ if every element of A is an element of B (and where it's possible that $A = B$). Transitivity and reflexivity are obvious, antisymmetry is true almost by definition since we know that $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

Note that $\subseteq$ is a relation on *sets*. A and B might be, say, *sets* of integers, but not integers. $\subseteq$ in this case is a relation on $2^{\underline{\mathbf{Z}}}$, not on $\underline{\mathbf{Z}}$.

We can use $\subseteq$ to order *some* sets of integers, e.g.

$\varnothing \subseteq \{5\} \subseteq \{5,7\} \subseteq \{2,3,5,7\} \subseteq \{\text{primes}\} \subseteq \underline{\mathbf{N}} \subseteq \underline{\mathbf{Z}}$

But not all sets are comparable! If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then it's not true that $A \subseteq B$ nor that $B \subseteq A$. These two sets are incomparable under $\subseteq$.

Definition: A *partially-ordered set*, or *poset*, is a set S together with a partial order on S. We write a poset as an ordered pair.

Examples: $(2^{\underline{\mathbf{Z}}}, \subseteq)$ is a poset. $(2^{\underline{\mathbf{N}}}, \subseteq)$ is another poset, as are $(\underline{\mathbf{N}}, |)$ and $(\underline{\mathbf{Z}}, \leq)$. But $(\underline{\mathbf{Z}}, |)$ is not a poset since "divides" is not antisymmetric on $\underline{\mathbf{Z}}$.

Hasse Diagrams

Let's take $S = \{1, 2, 3\}$ as our underlying set and consider the poset $(2^S, \subseteq)$, that is, the poset consisting of the subsets of S under the partial order $\subseteq$. Since there are only 8 subsets of S ($|2^S| = 2^3$, remember?) it's easy to write out the entire partial ordering. [Blackboard]

The picture on the blackboard is called a *Hasse diagram* it shows the (partial) order relationship between the various objects. We always draw a Hasse diagram so that if x R y then y is above x. More examples:

The Hasse diagram for the partial order $|$ on $\underline{\boldsymbol{N}}$.

The Hasse diagram for a total order is a vertical line. Note that it may or may not have a bottom or a top.

The Hasse diagram for the poset

({English words}, "is a prefix of")

which has disconnected pieces.

The Hasse diagram for the poset ({digits} , =), an extreme example of disconnected pieces.

{Max,min}imal Elements

Definition: Let P = (S,R) be a poset. An element x of S is called *minimal* if the only element y of S such that y R x is $y = x$. (More formal definition in Grimaldi.) Said another way, x is minimal if, in the Hasse diagram of P, there is no line going downwards from x.

Examples: The empty set is a minimal element of (2^S,□). 1 is a minimal element of (***N***, |). "are", "we", "having", and "fun"—but not "yet"—are minimal elements of ({words}, prefix of). (***Z***, <) has no minimal elements.

So a poset may have zero, one, or many minimal elements. Theorem: A finite poset always has at least one minimal element. Sketch of proof: Start anywhere and go downwards. In a finite poset you can't do this forever; when you have to stop, you've reached a minimal element. An infinite poset, even one that's not a total order, may not have a minimal element!

A *maximal* element of a poset is one which has no *upward* line in the Hasse diagram. Everything above applies to maximal elements: there may be zero, one, or many, but a finite poset must have at least one. Note that an element of a poset can be both maximal and minimal!

Tops and Bottoms

Let P = (S, R) be a poset. An element $x \in S$ is a *top* (Grimaldi: greatest element) for P if y R x for all $y \in S$. An element $\bot \in S$ is a *bottom* (Grimaldi: least element) for P if $\bot$ R y for all $y \in S$.
Intuitively: x is a top if you can follow lines downward from x to every element of S. Similarly for bottom.

Examples: S is a top of $(2^S, \subseteq)$ and $\varnothing$ is a bottom.
1 is a bottom for $(\underline{\boldsymbol{N}}, |)$ and this poset has no top.
({English words}, prefix of) has neither top nor bottom.

A poset can have a top, a bottom, both, or neither.
Even a finite poset needn't have a top or a bottom, and even an infinite poset—even a total order!—can have a top, a bottom or both. [Blackboard examples]

A top is always a maximal element, but a maximal element needn't be a top, not even if the maximal element is unique. But a unique maximal element in a *finite* poset is a top. Mutatis mutandis for bottom.

Theorem: A poset can have at most one top.
Proof: If x and y are tops, then x R y and y R x, so $x = y$ by antisymmetry. Same for bottoms, of course. Can an element of a poset be both a top and a bottom?

Bounds & Lattices

Suppose (S, R) is a poset and B is a subset of S. Then an element $x \in S$ is called a *lower bound* of B if x R b for every $b \in$ B, that is, if x is below every element of B. Similarly, $y \in S$ is an *upper bound* for B if y is above every element of B.

Note that an upper or lower bound must be comparable to every element of the set it bounds, but needn't be a member of that set. [Examples]

If x is a lower bound of B, then x is a *greatest lower bound (glb)* of B if y R x for any lower bound y of B. [Similar definition for *least upper bound, lub*.]

A set may have zero, one, or many lower bounds.
Even if it has lower bounds, it need not have a glb.
But it can't have more than one glb. [Easy proof]
Same for upper bounds and glb, of course. (Duality.)

A poset is called a *lattice* if every pair of elements has both a glb and an lub. [Examples]

Regrettably, we're not doing much with these concepts.

Topological Sort

Let (*S,* R) be a poset and let T be a total order on *S*. We say that R is embedded in T if for all *x, y* in *S* such that *x* R *y*, we also have *x* T *y*.

What does this mean? It means that T *preserves the partial order* created by R; if R says that *x* is below *y* then T says the same thing. Of course T may say more, since pairs of elements may be incomparable in R but are (perforce) comparable in T.

[Blackboard: ({1,2,3}, □) embedded in a total order.]

Topological sort is the process of embedding R in a total order on *S*, that is, finding a T in which R is embedded.

Why is TS important? Real-world example: Suppose tasks T_1, T_2, . . ., T_n must be performed respecting ordering constraints (usually called "dependencies"): e.g., T_2 depends on T_4, T_1 must precede T_7, etc. The constraints create a partial ordering of the tasks. We need to carry out the tasks according to some total order that respects the partial order of the dependencies.

[Set-theoretically, that is, regarding R and T as sets of ordered pairs, "*embedded in*" just means "*subset of*"!]

Sorting Topologically

Here's how to perform TS on a finite poset. Start with P = (S,R) and an empty total order Q = (∅, T).

[1] Find a maximal element of P; call it x

[2] Remove x from S. Technically, this means replacing S with $S-\{x\}$ and removing from R all ordered pairs containing x. Intuitively, it means erasing x from the Hasse diagram of P and also erasing all lines downward from x. (There aren't any lines upward.)

[3] Add x as the lowest element of Q. Intuitively, this means adding x just below the lowest element of Q and drawing a line from x up to the bottom element of Q. Technically it means adding x to Q and then adding to T a pair (x,y) for every y in Q.

[4] If S is empty, stop. Otherwise go back to step [1].

The idea behind the algorithm is simple: Build Q from the top, at each step adding some maximal element from the current S. The resulting total order is not unique. Proving this algorithm correct would be a good project.

Note that the algorithm satisfies a mathematician but not a computer scientist. As described, the algorithm runs in time $\Theta(n^3)$ since step 1 can be $\Theta(n^2)$. We can do better.

Relations as Matrices

An *0-1 matrix* is, surprisingly, a matrix whose entries are zeroes and ones. We can express a relation R on a finite set *S* as an 0-1 matrix M_R with $|S|$ rows and $|S|$ columns: Each row represents an element of *S* and each column likewise (in the same order!). The entry in row *x*, column *y* is 1 if (x,y) is in R and is 0 otherwise. [Blackboard example]

We can now express properties of R via its matrix M_R:

R is reflexive if and only if the main diagonal of M_R is all 1s; R is irreflexive if and only if that diagonal is 0s.

R is symmetric if and only if M_R is symmetric around its main diagonal. R is antisymmetric if and only if distinct elements that are "mirror images" in the main diagonal are never both 1.

R is transitive if and only if any element that is 1 in M_R^2 is 1 in M_R (but we're not doing matrix multiplication).

R is a total order if and only if there's a way to label the rows and columns such that M_R is all 1s on and above the main diagonal and is all 0s below.

Counting

Grimaldi loves to count kinds of relations on finite sets. Let's indulge him. Suppose S is a finite set with $|S| = n$.

How many relations are there on S? $|S \times S| = n^2$ and any subset is a relation. The number of subsets of a set of size n^2 is $2^{(n)(n)}$ because for each of the n^2 ordered pairs we have a two-way choice: in or out.

How many relations on S are reflexive? Now we have no choice about the n ordered pairs (x,x); they all must be *in* the relation. We can still choose about the other $n^2 – n$ pairs independently. So the answer is $2^{(n)(n–1)}$.

How many relations on S are symmetric? The ordered pairs *other* than (x,x) can be grouped into $(n^2–n)/2$ doubles (x,y) and (y,x). Each double must be in or out. The n non-doubles (x,x) can be in or out independently. So the answer is 2 to the power $n + (n^2–n)/2$.

How many relations on S are antisymmetric? Now for each double there are three choices: Both out, one in, or the other in. So the answer is $(2^n)(3^{(n)(n–1)/2})$.

Exercises: How many relations on S are irreflexive? How many are reflexive and symmetric? Reflexive and antisymmetric? Neither reflexive nor irreflexive?