MATH 22

Lecture T: 11/6/2003

EQUIVALENCE RELATIONS

. . . in the field of public education
the doctrine of "separate but equal"
has no place.

—Earl Warren, *Brown v. Board of Education of Topeka*

All animals are equal, but some
animals are more equal than others.

—George Orwell, *Animal Farm*

Administrivia

- http://denenberg.com/LectureT.pdf
- NO CLASS NEXT TUESDAY (Veteran's Day)
- NOTE CHANGE IN HW 10: 7.4 # 6 and 8
- Meet-the-professors immediately after this class
  (we'll try to quit slightly early if possible)
- Shortened office hours today (as if anyone cares)

Equivalence Relations

A binary relation on a set S is an *equivalence relation* if it is reflexive, symmetric, and transitive.

Examples:

"is a sibling of" [with our simplistic definition]

"is in the same state as" (a relation on US cities)

"has the same parity as" (i.e., both even or both odd)

"is the same age as"

"is equal to"

Note that these all examples have an underlying theme of "sameness". This is not a coincidence.

We often use the symbol $\equiv$ for an anonymous equivalence relation.

Another important example: Suppose $f : A \to B$ is any function, and define a relation $\equiv$ on A [not on B!] as follows: For $x,y \in A$, define $x \equiv y$ if $f(x) = f(y)$.
Then $\equiv$ is an equivalence relation, as we can check.
Fact: *Every* equivalence relation can be written this way!
E.g., $f : \mathbf{Z} \to$ {odd, even} by $f(n) = n$'s parity.

Equivalence Classes

Let S be a set and $\equiv$ an equivalence relation on S.
For $x \in S$ we define $[x]$, the *equivalence class of x*, as

$$[x] = \{ y \in S \mid x \equiv y \}$$

That is, $[x]$ is the set of all y in S that are equivalent to x.

Examples:

Under the "is in the same state as" relation, the equivalence class of Omaha is the set of all cities in Nebraska.

Under the "has the same parity as" relation, the equivalence class of 73 is the set of all odd integers.

Under the "is a sibling of" relation, the equivalence class of a person is the set of his or her siblings (including him- or herself, so the set is never empty).

Under the equality relation, every object is in an equivalence class by itself.

Fact: Any object x is a member of its own equivalence class. That is, $x \in [x]$ for any $x \in S$.
Proof: Obvious by reflexivity.
Corollary: An equivalence class is never empty.

Elementary Results

Now suppose $x \in S$ and $y \in [x]$, which means that $x \equiv y$. What is $[y]$, that is, which elements are equivalent to y?

Suppose $z \in [y]$, that is, $y \equiv z$. By transitivity, $x \equiv z$. So if $z \in [y]$ then $z \in [x]$, which is to say, $[y] \subseteq [x]$.

Now let $z \in [x]$, i.e. $x \equiv z$. Then $y \equiv x$ by symmetry, so by transitivity $y \equiv z$, i.e. $z \in [y]$. This proves $[x] \subseteq [y]$.

Putting these results together, we have shown that

if $y \in [x]$, then $[y] = [x]$.

Now let's go the other way: Suppose $[x] = [y]$. But we know that $y \in [y]$, so $y \in [x]$, so $x \equiv y$ by definition. Summarizing this whole mess gives:

$x \equiv y$ if and only if $[x] = [y]$

A critical Theorem: Suppose that $[x]$ and $[y]$ have any element at all in common, say z, so that $x \equiv z$ and $y \equiv z$. By symmetry and transitivity, $x \equiv y$, which means that $[x] = [y]$ by the previous result. We've shown that if $[x]$ and $[y]$ have any elements in common then they're equal. Said another way, $[x]$ and $[y]$ are either equal or disjoint.

Partitions

A *partition* of a set S is a finite or infinite collection $\{ S_1, S_2, \ldots \}$ of nonempty subsets of S such that

- The union of the all the S_i is equal to S, and
- The subsets are pairwise disjoint, that is, $S_i \cap S_j$ is empty for every pair i and j.

Intuitively, a partition chops S up into disjoint pieces that cover all of S. Each S_i is called a *cell* of the partition.

The Basic Theorem: If S is a set and $\equiv$ is an equivalence relation on S, then the equivalence classes of $\equiv$ are a partition of S. Moreover, any partition of S yields an equivalence relation on S.

We've proved part of this theorem already by showing that any two distinct equivalence classes are disjoint. Their union is all of S since any $x \in S$ is in one of them.

To prove the last part: Given a partition $\{ S_1, S_2, \ldots \}$ of S, define a relation $\equiv$ by $x \equiv y$ if x and y are in the same S_i. Show that $\equiv$ is an equivalence relation and that its equivalence classes are exactly the sets S_i. (Easy)

Examples

• There are 50 equivalence classes of the "is in the same state as" equivalence relation, one for each state.

• The "has the same parity as" equivalence relation has two equivalence classes, one containing all the *even* integers and one containing all the *odd* integers. So we have [2] = [8] = [2002] and [1] = [–99] = [221], etc.

• The equivalence classes of the equality relation are singleton sets. (This is an extreme example of a partition; the set S is chopped up as far as possible!)

• Dually, define an equivalence relation $\equiv$ as follows: $x \equiv y$ for *every* x and y in S! This relation has only a single equivalence class, namely all of S. This is the other extreme: set S is chopped up as little as possible.

• Let S be the set of 3-letter "words", i.e. permutations:

$$S = \{ \text{ABC, CAT, TCA, RSQ, QYY, AAA, } \ldots \}$$

Define an equivalence relation on S by $x \equiv y$ if x is a rearrangement of y. The equivalence classes are just the *combinations* of letters taken 3 at a time, and there are |S| / 6 such classes. I.e., "clumps" were really equivalence classes! This technique is used all the time.

An Important Example

The binary operator "mod" takes two integers and returns the remainder when the first is divided by the second (which must be positive). More formally, x mod y is the unique r such that $0 \le r < y$ and $x = qy + r$ for some integer q (Division Theorem). Examples:

10 mod 3 = 1	15 mod 6 = 3	12 mod 3 = 0
10 mod 1 = 0	7 mod 100 = 7	–1 mod 10 = 9

Note that "mod" is an operator (function), not a relation.

Now fix some $N > 0$ and define a relation as follows: $x \equiv y$ if $x \bmod N = y \bmod N$. (Another way to define the same relation is $x \equiv y$ if $N \mid x–y$.) This relation, known as "equals modulo N", is an equivalence relation. For example, suppose $N = 8$. Then we have, inter alia,

$$[1] = [9] = [–7] = \{ \ldots –15, –7, 1, 9, 15, 23, \ldots \}$$

$$[3] = [19] = \{ \ldots –13, –5, 3, 11, 19, 27, \ldots \}$$

$$[0] = [8888] = \{ \ldots –24, –16, –8, 0, 8, 16, 24, \ldots \}$$

If $N = 2$, we have the "same parity" relation, with two classes. In general, "equals mod N" has N equivalence classes, one for each remainder $\{ 0, 1, 2, \ldots, N–1 \}$.
If we permit reals instead of integers, we get infinitely many equivalence classes, one for each real in $[0,N)$.

[Refinements]

Suppose P = { S_1, S_2, . . . } is a partition of S.
A *refinement* of P is another partition { T_1, T_2, T_3, . . . } of S with the property that each T_i is a subset of some S_j.

Intuition: A refinement Q of a partition P is just a *finer* chopping-up of S, one that *respects the boundaries* of P. Each cell S_j of partition P is either a cell of Q or is itself partitioned into cells. So no cells of Q cross the cell boundaries of P!

Example: Let R be the relation "equals mod 4" on ***Z***. The partition induced by this relation has 4 cells, one for each remainder 0, 1, 2, 3. Now let R_2 be the relation "equals mod 12". This relation induces a partition with 12 cells, *each of which is a subset of a cell of R*.
(For example, [10] of R_2 is a subset of [2] of R.)
But the partition induced by "equals mod 1001", though it has lots more cells, is not a refinement of R.

Example: The relation = induces a partition that refines *any* other partition (most extreme chopping-up).

Geometric Examples

Let S be the set of points on the line, and define $x_1 \equiv x_2$ if $\lfloor x_1 \rfloor = \lfloor x_2 \rfloor$. What are the equivalence classes of this relation? How many equivalence classes are there?

Let S be the set of points on the line, and define $x_1 \equiv x_2$ if $\lfloor 2x_1 \rfloor = \lfloor 2x_2 \rfloor$. What are the equivalence classes of this relation? How many are there? [It's a refinement.]

Let S be the set of points in the plane, and define $z_1 \equiv z_2$ if z_1 and z_2 are the same distance from the origin. What are the equivalence classes of this relation? How many are there?

Let S be the set of points in the plane, and define $z_1 \equiv z_2$ if z_1 and z_2 are less than one unit apart. What are the equivalence classes, and how many are there?

Let S be the set of points in the plane, and define $(x_1,y_1) \equiv (x_2,y_2)$ if corresponding coordinates have the same sign, that is, if x_1 and x_2 are both either positive, negative, or zero, and the same for y_1 and y_2. What are the equivalence classes, and how many are there?