MATH 22

Lecture V: 11/18/2003

HAMILTONIAN GRAPHS

All communities [graphs] divide themselves into the few and the many [i.e., are bipartite].

—Alexander Hamilton, *Debates of the Federal Convention*

Before a group [graph] can enter the open society, it must first close ranks.

—Charles Vernon Hamilton, *Black Power!*

Truth, like a torch.

—Sir William Hamilton, *On Math 22*

Administrivia

- http://larry.denenberg.com/math22/LectureV.pdf
  (note change!)

- Comments on grading

- Comments on Project 6 #1 and #2

- EXAM 3: MONDAY 11/24
  Review Thursday; send questions via email (hah!)
  Note change in coverage (11.5 is out)

Today: Hamiltonian graphs

Hypercubes

The *n-dimensional hypercube*, Q_n, is the undirected graph whose vertices are the bit strings of length *n* and with edges connecting vertices that differ in a single bit. [blackboard examples clarifying this opaque definition]

Properties of the hypercube:

- Q_n has 2^n vertices.
- Q_n is regular of degree *n*.
- Q_n has $n2^{n-1}$ edges. [Proof: By the first two properties, the sum of the node degrees is $n2^n$. Cut this in half.]
- Q_n is connected, and between any two nodes there is a path of length at most *n,* with mean *n*/2. [Digression on processor connectivity; the distance between processors in a hypercube grows as the base-2 log of their number.]
- Q_n can be constructed from two copies of Q_{n-1}: Connect each pair of corrresponding nodes with an edge, add “0” to the node labels of one copy, “1” to the other
- Q_n is isomorphic to the Hasse diagram of the partial order □ on the set of subsets of $\{1, 2, 3, ..., n\}$
- Q_n, properly drawn, is a perspective view of the *n*-cube, an object with 2*n* bounding faces of dimension *n*–1

Hamiltonian Graphs

A path in a directed or undirected graph is called *Hamiltonian* if it visits each vertex exactly once.

A cycle in a graph is called *Hamiltonian* if it visits each vertex exactly once (ignoring the fact that the start vertex is also the final vertex).

A graph is called *Hamiltonian* if it has a Hamiltonian cycle, and (less commonly) *traceable* if it has a Hamiltonian path.

[blackboard examples]

There is no known efficient way to determine whether a graph is Hamiltonian (but it hasn't (yet) been proved that the problem is inherently inefficient!).

We first investigate some properties that a Hamiltonian graph must have, then we show some special graphs for which we can prove Hamiltonianicity. All these proofs have the following basic form: If a graph has enough edges, it must be Hamiltonian!

Necessary Conditions

Every undirected Hamiltonian graph is connected, and every directed Hamiltonian graph is strongly connected.

Every vertex of a Hamiltonian graph has degree ≥ 2.

If v is a node in an undirected Hamiltonian graph G and $\deg(v) = 2$, both edges incident on v are part of every Hamiltonian cycle of G.

[blackboard examples]

Definition: An undirected graph $G = (V,E)$ is *bipartite* if V can be written as a union $V_1 \cup V_2$ such that for each edge $\{x,y\} \in E$ we have $x \in V_1$ and $y \in V_2$ or vice versa. That is, G is bipartite if its vertices can be divided into two sets where every edge goes *between* the sets and there is no edge *within* a set. [blackboard examples]

If G is bipartite and Hamiltonian, then the two parts of G must have the same number of vertices.

Sufficient Conditions I

Every complete undirected graph is Hamiltonian.

Proof: Just number the vertices any old way, and consider the cycle $v_0, v_1, \ldots, v_n, v_0$. All edges exist.

A *tournament* is a directed graph (V,E) that is a complete undirected nonmultigraph if you "erase the arrowheads"; i.e., for every distinct pair of vertices $x, y \in V$, exactly one of (x,y) and (y,x) is in E. [blackboard examples]

Every tournament has a (directed) Hamiltonian path.

Proof: We will show how to take any path and add a new node to the path; we can then start with any 1-edge path and repeat this process until every node is added.

So suppose the path is $x_1, x_2, \ldots, x_m$ and node y isn't yet in the path. If edge (y,x_1) exists, add y before x_1 and we're done. Otherwise, edge (x_1,y) exists. If edge (y,x_2) exists, add y between x_1 and x_2 and we're done. Otherwise edge (x_2,y) exists. Continue in this way; if edge (y,x_j) exists add y to the path between x_{j-1} and x_j. And (y,x_j) doesn't exist for any j, then edge (y,x_m) doesn't exist, so edge (x_m,y) exists and we can add y to the end of the path. QED.

Sufficient Conditions II

If $G = (V,E)$ is an undirected, loop-free graph, and if for every pair of distinct nodes $x,\ y \in V$ it is true that

$$\deg(x) + \deg(y) \geq n - 1$$

where $n = |V|$, then G has a Hamiltonian path.

Lemma: G is connected. For if not, pick nodes x and y in different connected components of G. Node x has degree at most c_x-1, where c_x is the number of nodes in its component; similarly, $\deg(y) \leq c_y-1$. So

$$\deg(x) + \deg(y) \leq c_x + c_y - 2 \leq n - 2$$

contradicting the hypothesis $\deg(x)+\deg(y) \geq n-1$.

Just as before, we'll show how to add a new node to any simple path to make a longer simple path; then we can start with any single edge (i.e., simple path of length 1) and extend it to a Hamiltonian path.

So suppose we have a simple path $x_1, x_2, \ldots, x_m$.
If x_1 is adjacent to any node y not already on the path, we can add y before x_1. If x_m is adjacent to any node z not on the path, we can add z after x_m. We need to show how to extend the path when both of x_1 and x_m are adjacent *only* to nodes already on the path.

(Proof continued)

Claim: In the case we're considering, the graph has a cycle on the vertices $x_1, x_2, \ldots, x_m$ already in the path, though not necessarily in this order. (We'll prove this claim shortly.) Now let y be a vertex not already in the path. Since G is connected, there is a path from G to some vertex x_p in the path. So we can add y to the path by constructing a new path starting at y, going to x_p, around the cycle to x_{p-1}, and we're done.

We still have to prove that the cycle exists! If x_1 and x_m are adjacent, we're done. Even if not, if there's any p such that x_1 is adjacent to x_p and x_{p-1} is adjacent to x_m, we're done. [blackboard picture] The only way there can be a problem is if there is no such x_p. But we can show that this can't happen, by contradiction.

So suppose that x_1 is adjacent only to x_2 and to some of the vertices $x_3, \ldots, x_{m-1}$, and that x_m is adjacent only to x_{m-2} and some vertices $x_1, \ldots, x_{m-3}$. For each vertex that x_1 is adjacent to, there's a vertex that x_1 can't be adjacent to, since there is no p such that x_1 is adjacent to x_p and x_m is adjacent to x_{p-1}. So together, x_1 and x_m are adjacent to at most $m-1$ vertices, which is less than $n-1$ (since y, at least, isn't on the path), contradicting the hypothesis.

Sufficient Conditions III

If the degree of each vertex of an undirected loop-free graph G is at least $(n-1)/2$, where n is the number of vertices of G, then G has a Hamiltonian path.

[Proof obvious via the preceding theorem.]

If G is undirected and loop-free with $n \geq 3$ vertices, and if $\deg(x) + \deg(y) \geq n$ for all nodes x and y that are *not* adjacent, then G has a Hamiltonian cycle. [Proof in Grimaldi]

If G is undirected and loop-free with $n \geq 3$ vertices, and if $\deg(x) \geq n/2$ for all vertices x, then G has a Hamiltonian cycle. [Proof obvious from the preceding.]

If G is undirected and loop-free with $n \geq 3$ vertices and at least $C(n-1,2)+2$ edges, then G has a Hamiltonian cycle. [Proof in Grimaldi. Intuition: For nonadjacent vertices x and y, G has at most $C(n-2, 2)$ edges that don't touch either one. Then since G has $C(n-1,2)+2$ edges total, a little arithmetic shows that x and y must together touch at least n edges, so a preceding theorem applies.]

Weighted Graphs; TSP

A *weighted graph* is one in which real numbers, called *weights,* are attached either to the nodes or edges or both. Weights can be used to model lots of things.

For example, let G be a graph whose vertices are cities, with an edge between two cities if there is a nonstop flight between them. The weight of each edge is the cost of the flight.

The *Travelling Salesperson Problem, TSP*, is as follows: Given a complete undirected graph with weighted edges, find the Hamiltonian cycle with least total weight.

[Blackboard example. Note that the graph need not be complete since if we want to leave out any edge we just assign it an infinite weight.]

No efficient solution to TSP is known, though it has not been proven that no efficient solution exists. Resolving this question guarantees you will be prominently named in every book on algorithms from now until the end of time. Better get started.