MATH 22

Lecture W: 11/20/2003

REVIEW

Nature fits all her children with
something to do;
He who would write and can't
write, can surely review.
—James Russell Lowell,
A Fable for Critics

Administrivia

- http://larry.denenberg.com/math22/LectureW.pdf

- EXAM 3: MONDAY 11/24
  11:50 AM – 1:20 PM
  LOCATION: ROBINSON 253

- Grading policy

What You Need

- Number Theory
  - Divisibility and divisors
  - Primes and composites
  - The Division Theorem and base conversion
  - GCD and the Euclidean algorithm
  - The Fundamental Theorem of Arithmetic and applications

- Relations
  - Properties of relations
  - Partial orders, Hasse diagrams, terminology
  - Equivalence relations and partitions

- Graphs
  - Directed vs. undirected graphs
  - [Terminology: path, circuit, trail, etc.]
  - Vertex degree, edge counting theorem
  - Connectivity, complement, completeness
  - Subgraphs, graph isomorphism
  - Eulerian circuits and Euler's Theorem

Number Theory

If you successively divide integer *N* by integer $b > 1$, always throwing away the remainder, the succession of remainders (read in reverse order) is the representation of *N* in base *b*.

Example: Express 12345 in base 2.

The *GCD* of two integers *a* and *b* is the largest number that divides both, that is, if $x = \gcd(a,b)$, then $x \mid a$ and $x \mid b$ and there is no $y > x$ such that $y \mid a$ and $y \mid b$.

If you take integers *a* and *b*, and successively divide the larger by the smaller, always replacing the larger with the remainder of the division (and throwing away the quotient), then the smallest number you get to (just before getting to zero) is the GCD of *a* and *b*. This is *Euclid's algorithm*.

Example: gcd(12345, 345) = gcd(345, 270)
= gcd(270, 75) = gcd(75, 45)
= gcd(45, 30) = gcd(30, 15) = 15

Number Theory

Find gcd(n,n+1) and lcm(n,n+1). What are the possible values for gcd(n,n+2) and gcd(n,n+3)?

gcd(n,n+1) is obviously 1; no larger number can divide both n and n+1! But if you insist on a proof, here goes: If x divides both a and b, then it divides $a-b$. Now suppose $y = \gcd(n,n+1)$, and therefore we have both $y \mid n$ and $y \mid (n+1)$. Then y divides the difference between n and $(n+1)$, which is 1. But the only number that divides 1 is 1 itself, so $y = 1$.

If $z = \gcd(n,n+2)$, then $z \mid n$ and $z \mid (n+2)$, so z divides their difference, which is 2. The only possibilities are z=1 and z=2 (these are the only numbers that divide 2). But z could be either one.

Similarly, the possible values for gcd(n,n+3) are 1 and 3.

The easiest way to get lcm(n,n+1) is to recall the duality

$$\gcd(a,b)\mathrm{lcm}(a,b) = ab$$

The gcd of n and n+1 is 1, so their lcm must be $n(n+1)$.

Number Theory

How many positive divisors are there of the number

$$N = 2^{14}3^{9}5^{8}7^{10}11^{3}13^{5}37^{10}$$

Any divisor of N has 0, 1, 2, 3, . . ., or 14 factors of 2, multiplied by 0, 1, 2, . . ., or 9 factors of 3, multiplied by 0, 1, 2, . . ., or 8 factors of five, etc. (For example, the number $2^{9}3^{4}7 11^{3}37^{3}$ is a divisor of N, note that it contains 0 factors of 5 and also 0 factors of 13.)

So to choose a divisor of N, there's a 15-way choice for the number of factors of 2, a 10-way choice for the number of factors of 3, etc. The answer is therefore

$$(15)(10)(9)(11)(4)(6)(11)$$

The Division Theorem: Given any two integers a and b with b positive, there are unique integers q and r such that $a = qb + r$, with $0 \le r < b$.

Relations

Reflexive, symmetric, antisymmetric, transitive. Partial orders and Hasse diagrams. Equivalence relations, partitions, and equivalence classes.

Describe geometrically the equivalence classes and partition of $\underline{\boldsymbol{R}}^2$, the plane, induced by the relation R where (x_1,y_1) R (x_2,y_2) if $x_1 = x_2$.

Answer: Two points are equivalent if they have the same x-coordinate. So all points on a single vertical line are equivalent. Each equivalence class consists of a single vertical line (infinite both ways); the plane is partitioned into an infinite number of vertical lines.

If $|A| = 4$, how many relations on A are reflexive? Symmetric? Reflexive and contain 5 pairs? Symmetric and contain a particular pair? Antisymmetric and symmetric? Reflexive, symmetric, and antisymmetric?

All such problems should be attacked by thinking of relations as 0-1 matrices and counting matrices. Reflexive: main diagonal all 1. Symmetric: Mirror image entries equal. Antisymmetric: Mirror image entries not both 1. [The "mirror" is the main diagonal.]

Relations

Let S_n denote the set of all connected subgraphs of K_n. List the elements of S_4. Define the partial order R on S_n as G_1 R G_2 if G_1 is a subgraph of G_2. Draw the Hasse diagram for the poset (S_4,R). [blackboard]

Let $X = \{0,1,2\}$ and $A = X \times X$. Define relation R on A by (a,b) R (c,d) if either (i) $a < c$, or (ii) $a = c$ and $b \leq d$. Prove that R is a partial order. Draw the Hasse diagram.

To prove R is reflexive, we must prove that (x,y) R (x,y) for all $x,y \in X$. But (x,y) R (x,y) if $x < x$ [not true], or if $x = x$ and $y \leq y$ [true!]. So R is reflexive.

To prove R is antisymmetric, we must prove that

if (x,y) R (z,w) and (z,w) R (x,y), then $(x,y) = (z,w)$

If (x,y) R (z,w), then it could be that $x < z$. But if so, then it can't be that (z,w) R (x,y) since we wouldn't have either $z < x$ or $z = x$! So it must be that $x = z$ and $y \leq w$. In this case, the only way that we can have (z,w) R (x,y) is if $w \leq y$. But if $y \leq w$ and $w \leq y$, then $y = w$. So we've shown $x = z$ and $y = w$, QED.

[This p.o. is called *lexicographic*, or *dictionary*, order.]

Graphs

Euler's Theorem: An undirected graph or multigraph has an Eulerian circuit if and only if each of its vertices has even degree. An undirected graph or multigraph has an Eulerian trail if and only if it has exactly two vertices with odd degree.

Graph edge count theorem: The sum of the vertex degrees of a graph, taken over all vertices of the graph, is equal to twice the number of edges in the graph.

Zillions of definitions: vertex, node, degree, in-degree, out-degree, regular, adjacent, connected, strongly connected, complement, complete graph, subgraph, isomorphic graph, plus walk, trail, path, circuit, cycle,. . .

Graphs

How many edges does K_n have?

There is an edge between every pair of nodes, so there's one edge for each way to select a pair of nodes. The answer is $C(n,2)$.

Let S_n denote the set of all connected subgraphs of K_n. List the elements of S_4. Draw the Hasse diagram for the poset (S_4,R) where R is the relation "is a subgraph of".

[answer on blackboard]

If G has n vertices and is isomorphic to its complement, how many edges must it have? Find a graph on 4 nodes that is isomorphic to its own complement, and one on 5 nodes. Prove that no such graph on 6 nodes can exist.

If G has n nodes and e edges, then its complement has $C(n,2)-e$ edges. If G and its complement have the same number of edges e, then $2e = C(n,2)$ or $e = (n)(n-1)/4$. A self-complementary graph with 4 vertices has 3 edges and one with 5 vertices has 5 edges. But a self-complementary graph on 6 vertices would have to have 7.5 edges and so can't exist.

Graphs

Let G_n be the loop-free undirected graph whose vertex set is the set of all n-tuples of 0s and 1s, with an edge between two nodes if they differ in exactly two places. How many connected components does G_n have?

[blackboard drawing of G_3]

Say that a node has *even parity* if it has an even number of 1s, and *odd parity* otherwise. So, e.g., 0011010 has odd parity. Clearly every edge of G_n connects two nodes of the same parity. So there is never any path from any node of odd parity to one of even parity. This means that G_n has at least two connected components.

Could it have more? That is, is there a path between any two nodes of even parity, and between any two nodes of odd parity? That is, can we take one even-parity bitstring and change it into any other by switching two bits at a time? Sure: Just switch all the bits off (two at a time), then switch on the ones you want (two at a time). A similar trick works for odd parity. So $\square(G_n) = 2$ for all n.

Graphs

Find the number of edges in Q_8, the 8-dimensional hypercube. Find the length of a longest path in Q_8.

The first part is trivial if you remember the edge-count theorem! Every vertex in Q_8 has degree 8, and there are 2^8 vertices. So the total vertex degree is $8(2^8) = 2^{11}$. The number of edges must be half of this, namely 2^{10}. Similarly, Q_n has $n(2^n)/2$ edges.

For Grimaldi, a "path" is a walk that doesn't repeat any vertices (and, perforce, doesn't repeat edges). How far can we walk in the hypercube? Answer: It's not too hard to see that we can make a path on all vertices of Q_n.
Here's an inductive proof: Clearly you can make a path on Q_0. Now assume there's a path on Q_n, we need to make a path on Q_{n+1}. But recall that Q_{n+1} is two copies of Q_n combined! We can start anywhere in the first copy, walk a path covering all the nodes in the first copy (by the inductive hypothesis), follow an edge to the second copy, and walk a path covering all nodes in that copy. So the answer to the problem is 2^8!

(Actually, the path can be made to be a cycle if $n > 1$.)