MATH 22

Lecture X: 11/25/2003

PLANAR GRAPHS & EULER'S FORMULA

Regions of sorrow, . . . where peace and rest can never dwell, hope never comes.

—Milton, *Paradise Lost* (on Tufts)

Ornate rhetoric taught out of the rule of Plato.

—Milton, *Of Education* (on Math 22B)

Administrivia

- http://larry.denenberg.com/math22/LectureX.pdf
- Exam followup?
- Grading policy followup?
- N.B.: Final homework and projects are due on 12/4

Today: Planar graphs, Euler's Formula, and some consequences and generalizations.

Planarity

An undirected graph G, which can be a multigraph and can have self-loops, is *planar* if it can be drawn in the plane in such a way that no edges cross (this is always what "drawn in the plane" means). Note that edges don't have to be drawn as straight lines.

As definitions go, this one is rotten, but a more precise definition isn't really illuminating. Only undirected graphs today, by the way.

Any number of multiple edges or self-loops, no matter how many, don't really matter; they can always be drawn small enough not to affect anything.

[blackboard examples of definitely-planar and maybe-nonplanar graphs]

Question: How do we know if a graph is planar or not? If we can draw it in the plane, it's planar for sure. But if we can't, maybe we're just not clever enough. How do we know when to quit trying? [This is typical: It's easier to show how to do something than to prove there's no possible way to do it.]

Regions

When a graph is drawn in the plane, it divides the plane into regions called (surprisingly) *regions*. Sometimes they're called *faces* for a good reason that we'll see soon.

Note that there is always exactly one infinite region since a finite graph must occupy a finite portion of the plane. The term "region" is not defined for nonplanar graphs or for nonplanar drawings of planar graphs.

When a graph G is drawn in the plane, how many regions does it have? Can it depend on how you draw it? The answer is that the number of regions depends only on the graph and not on any particular drawing. It's a property of G, just like the number of vertices or edges.

Theorem: Suppose that G is a planar graph that has v vertices, e edges, and $\square$ connected components. Then any planar drawing of G has $\square + e - v + 1$ regions.

This Theorem is more usually written like this:

$$v - e + r = \square + 1$$

When G is connected, we get *Euler's Formula*

$$v - e + r = 2$$

of which we'll see two Awesome Generalizations later.

Proof of Euler's Formula

The proof is by induction on e, the number of edges.

Base case, $e = 0$: An edgeless graph on v vertices has v connected components, and no matter how you draw it there's only 1 region. Plug in and verify the Formula.

Inductive case: Suppose the Formula is true for all graphs with e or fewer edges, and let G be any graph with e+1 edges. Pick any edge e_0 from G and delete it, obtaining graph H, in which the Formula holds by the Inductive Hypothesis. There are two possibilities:

Possibility 1: e_0 joins two nodes in *distinct* connected components of H. In this case G has one fewer connected component and one more edge than H, but the number of regions remains the same [picture].
So the Formula is valid for G.

Possibility 2: e_0 joins nodes in *the same* connected component of H. In this case G has the same number of connected components as H, but has one more edge and one more region, so again the Formula is valid for G.
[Blackboard: semi-handwaving proof that G has one more region than H. Special case if e_0 is a self-loop!]

Corollaries of Euler's F

If G is a planar graph, not a multigraph, with no self loops and at least 2 edges, then $3r \leq 2e$. (That is, *you can't have lots of regions without lots of edges*!)
Proof: This is obvious when $r = 1$. If $r > 1$, then each region is bounded by at least 3 edges, so the total number of "edge-boundings" (instances where an edge touches a region) is $\geq 3r$. But each edge bounds at most 2 regions, so the number of "edge-boundings" is also $\leq 2e$. This implies that $3r \leq 2e$. [Where did we use the fact that G has neither self-loops nor multiple edges?]

Under the same hypotheses, $e \leq 3v - 3\square - 3$.
(That is, *a planar graph can't have too many edges*.)
Proof: From Euler's formula,

$$\square + 1 \ = \ v - e + r \ \leq \ v - e + 2e/3 \ = \ v - e/3$$

from which a little arithemetic gives the result. When the graph is connected, we have $e \leq 3v - 6$.

Every planar graph without self loops or multiple edges has at least one vertex with degree ≤ 5.
Proof: Assume there is a graph where all nodes have degree at least 6. The sum of the node degrees is then at least $6v$. This sum is also $2e$ (remember?) so we have $2e \geq 6v$ or $e \geq 3v$. But this contradicts $e \leq 3v - 3\square - 3$.
Corollary: K_n is non-planar for every $n \geq 7$.

Two Non-Planar Graphs

Fact: K_5, the complete graph on 5 nodes, is nonplanar.

Proof: In K_5 we have $v = 5$ and $e = 10$, so we don't have $e \leq 3v-6$ as we must in all connected planar graphs.

Definition: For integers $u,v \geq 1$, the *complete bipartite graph on u and v nodes*, written $K_{u,v}$, is the undirected bipartite graph with no multiple edges, with u vertices in one part and v vertices in the other part, and with every possible edge between the parts. [blackboard examples]

Fact: $K_{3,3}$ is nonplanar.

Proof: If $K_{3,3}$ were planar, it would have 5 regions by Euler's formula. Now $K_{3,3}$ contains no triangles [why?] so each region would touch at least 4 edges, making at least 20 "edge-boundings". But $K_{3,3}$ has only 9 edges and each bounds at most 2 regions, a contradiction.

It turns out that these two nonplanar graphs are more than just examples; they are the very essence of nonplanarity. The next theorem, which we won't prove, says essentially that every nonplanar graph contains one of these two graphs as (sort of) a subgraph!

Kuratowski's Theorem

Definition: Two graphs are *homeomorphic* if they can be made isomorphic via the following operations:

- Delete any edge $\{u,v\}$, then add a new vertex w and edges $\{u,w\}$ and $\{w,v\}$
- Delete any vertex w of degree two, delete (perforce) its adjacent edges $\{u,w\}$ and $\{w,v\}$, then add new edge $\{u,v\}$

Informally, two graphs are homeomorphic if they can be made the same by putting dots on lines, or erasing dots that have only two lines coming out of them and merging the lines. The phrase "*isomorphic up to vertices of degree 2*" means the same thing. [Blackboard examples]

Theorem: A graph is nonplanar if and only if it contains a subgraph homeomorphic either to K_5 or to $K_{3,3}$.
Proof: Half the proof is easy, since a graph with a subgraph homeomorphic to K_5 or $K_{3,3}$ clearly can't be planar. (This can't be plainer.) The converse is harder. This Theorem is called *Kuratowski's Theorem.*

[Blackboard examples of nonplanar graphs]

Polyhedra

A *polyhedron* is a solid object bounded by planar polygonal surfaces, each of which is called a *face*.

The vertices and edges of the faces are also the vertices and edges of the polyhedron.

[blackboard drawings, insofar as I can draw polyhedra]
Note that polyhedra need not be convex.

Every polyhedron corresponds to a connected planar graph with the same number of edges and vertices. To get this graph, puncture any face of the polyhedron and pull the hole open until the polyhedron is stretched out flat. [even less coherent drawings; artist needed!]

Each face of the polyhedron corresponds to a region of the planar graph; the face we punctured becomes the infinite region. It follows that $v - e + f = 2$ for any polyhedron, where f is the number of its faces.

A convex polyhedron is a *Platonic solid* if its faces are identical regular polygons, and at each vertex the same number of faces come together at the same (solid) angle.
[blackboard drawing of at most two Platonic solids]
Question: How many Platonic solids exist?

All the Platonic Solids

Suppose a Platonic solid has v vertices, e edges, and f faces. Suppose each face is a regular m-gon and that n faces meet at each vertex.

Fact:	$m \geq 3$	(each face must be at least a triangle)
Fact:	$n \geq 3$	(each vertex lies in at least three faces)
Fact:	$2e = mf$	(each edge lies in two faces, each of which has m edges)
Fact:	$2e = nv$	(there are $2e$ edge endpoints, and each vertex is the endpoint of n edges)

Using these facts and Euler's formula, we get:

$$2 = v - e + f = 2e/n - e + 2e/m = (2m - mn + 2n)e/mn$$

Therefore $(2m - mn + 2n) > 0$, hence $(mn - 2m - 2n) < 0$, hence $(mn - 2m - 2n + 4) < 4$, hence finally $(m-2)(n-2) < 4$.

But m and n are integers each at least 3, so the only possibilities are:

$m = n = 3$	(so e=6, v=f=4), the *regular tetrahedron*
$m = 4,\ n = 3$	(so e=12, f=6, v=8), the *cube*
$m = 3,\ n = 4$	(so e=12, f=8, v=6), the *octahedron*
$m = 5,\ n = 3$	(so e=30, f=12, v=20), the *dodecahedron*
$m = 3,\ n = 5$	(so e=30, f=20, v=12), the *icosahedron*

Awesome Generalization 1

Suppose P is a solid object bounded by plane polygonal faces, but is not a polyhedron because it has "holes".

[Blackboard drawing of a cube with a square hole through it. Note that each face must be a polygon, so the front "face" is really four faces, each a trapezoid, with edges between them. The object has 16 faces, 16 vertices, and 32 edges; it's regular of degree 4.]

The number of holes in P is called the *genus* of P. The object drawn on the blackboard has genus 1.

Theorem: Given any such solid object with v vertices, e edges, f faces, and genus g, we have

$$v - e + f = 2 - 2g$$

Proof: Omitted.

When the object is a polyhedron there are no holes, so $g = 0$ and the theorem reduces to Euler's formula.

Awesome Generalization 2

The object that generalizes vertex (0 dimensions), edge (1d), polygon (2d), and polyhedron (3d) to an arbitrary number of dimensions is called a *polytope*.

A polygon is bounded by vertices and edges.
A polyhedron is bounded by vertices, edges, and faces.
Similarly, an n-dimensional polytope is bounded by vertices, edges, faces, polyhedra, ..., and polytopes of all dimension up to $n-1$. We also consider an n-d polytope to be bounded by a single n-d polytope, namely itself.

Example: The *4-hypercube* is a 4 dimensional polytope bounded by 16 vertices, 32 edges, 24 (square) faces, 8 (cubical) polyhedra, and itself. [blackboard drawing]

Theorem: Let P be an n-dimensional polytope bounded by b_k polytopes of dimension k, for each $0 \le k \le n$. (So P has b_0 vertices, b_1 edges, etc., and $b_n = 1$.) Then

$$b_0 - b_1 + b_2 - b_3 + \ldots + (-1)^n b_n = 1$$

Proof: You must be joking. Check it on the 4-hypercube.

In 3 dimensions, this is $v - e + f - 1 = 1$, Euler's formula.
In 2d, it says any polygon has as many vertices as edges.
In 1d, it says that an edge joins two vertices.

Duality Again

Suppose G is a planar graph or multigraph, possibly including self-loops, and we have a planar drawing of G. A *dual graph of G*, written G^d, is generated from this drawing as follows:

- Each region in the drawing becomes a node of G^d.
- For each edge e of G we make an edge of G^d:
  If e separates two regions of G, we put an edge between the nodes of G^d that correspond to those regions. If e lies wholly within a region of G, we give the corresponding node of G^d a self loop.

A dual graph of G has as many vertices as G has regions, and has the same number of edges as G.

A dual graph of G is connected whether or not G is connected. But if G is connected, it follows from Euler's formula that G^d has as many regions as G has vertices.

A dual graph of G may be a multigraph and may have self loops, even if G is not a multigraph and has no self-loops.

A dual is generated from a *drawing* of G. Depending on how G is drawn, we may get different results. That is, a graph can have non-isomorphic dual graphs.