MATH 22

Lecture Y: 12/2/2003

GRAPH COLORING

Here of a Sunday morning
My love and I would lie
And see the colored counties
And hear the larks so high
About us in the sky.

—A.E.Housman,
A Shropshire Lad

Administrivia

- http://larry.denenberg.com/math22/LectureY.pdf

- Final project and problem set due next time, 12/4, at the *START* of lecture! [That word again is START.]

- FINAL EXAM THURSDAY DECEMBER 11, 8:30 – 10:30 AM, ROBINSON 253 (same as Exam 3)

- Next time:
  - Final review: bring or email questions!
  - Course evaluation: Bring a #2 pencil or blue or black ink pen, plus comments on the lecture notes. http://larry.denenberg.com/cs124-eval contains an example of review comments.

- Article for your consideration: "The Role of Logic in Teaching Proof" by Susanna S. Epp, American Math. Monthly vol 110 no. 10, December 2003, p 886.

Today: Vertex coloring and the Four Color Theorem

Vertex Coloring

Let G be an undirected graph without self-loops or multiple edges (this is the only kind of graph for today). To perform a *vertex coloring* of G, we assign a color to each node according to the following rule: *if two nodes are adjacent* (i.e., if they're connected by an edge) *then they must get different colors*. [blackboard examples]

Formally, a vertex coloring of G is a mapping f from the vertex set of G to any set C of objects called *colors*, such that if (u,v) is any edge of G then $\mathrm{f}(u) \neq \mathrm{f}(v)$.

Obviously we can color any graph using a different color for each vertex. The trick is to use as few colors as possible! Reason: Coloring is used to model situations with entities (nodes) that must be kept separate (edges); colors then model resources assigned to nodes.

Example: N committees must meet each week. If two committees have members in common, they must meet at different times. How many meeting times are needed? Answer: Let each committee be a node, with edges between nodes if the associated committees have common members. The number of times needed is the minimum number of colors required to color this graph.

Chromatic Number

Let G be an undirected loop-free graph. The *chromatic number of G*, written $\chi(G)$, is the smallest number of colors that can be used to perform a vertex coloring of G.

Examples:

$\chi(K_n) = n$, since no two nodes of K_n can have the same color (all possible edges exist).

If G is any edgeless graph, then $\chi(G) = 1$, since all nodes can get the same color.

If G is any bipartite graph, then $\chi(G) = 2$: we can color each part with a single color since no edge can touch two nodes in the same part. (Indeed, $\chi(G) = 2$ could serve pretty well as the *definition* of bipartite, with a little weirdness about what to do with edgeless graphs.)

Theorem: If the largest node degree in G is k, then the chromatic number of G is at most $k+1$.

Proof: Color the vertices of G one by one. With $k+1$ colors available you can't ever run into a problem, because no node is adjacent to more than k other nodes.

Planar maps and 4CT

When coloring a map, we wish to have adjacent regions colored with distinct colors. (Two regions are *adjacent* if they touch along a line segment, not just at a finite number of points.) This problem is the same as graph coloring. Here's why:

Given a map, construct a graph G with one vertex for each region and an edge connecting each pair of vertices that correspond to adjacent regions. Then $\chi(G)$ is the number of colors needed to color the map. [picture]

Theorem (Appel and Haken, 1976): If G is a planar graph, then $\chi(G) \leq 4$. Equivalently, four colors suffice to color any planar map. (We'll prove a simpler version.)

This theorem was first proposed in 1852 by a student who was coloring a map of the counties of England.

Applies also to maps on a sphere, which can be made planar by puncture-and-flatten. Not applicable to maps on a torus or more complex surfaces. (The number of colors necessary to color a map on a surface of genus g is $\lfloor(7 + \sqrt{(1 + 48g)}) / 2\rfloor$, which is 7 for a torus.)

5CT

We'll talk later about how to prove 4CT. But it's fairly easy to prove that 5 colors always suffice!

Theorem: If G is a planar graph, then $\chi(G) \leq 5$.

Proof: By induction on the number of vertices of G.

Base case: If G has 1 vertex then obviously G can be colored with 5 colors. (You may have noticed that this base case works perfectly well up to 5 vertices.)

Inductive case: Suppose all planar graphs with n vertices are 5-colorable, and let G be a planar graph with n+1 vertices.

By a corollary to Euler's formula (proved last time) we know that G has some vertex x with degree ≤ 5.

If x has degree four or fewer, we're done: Remove x from G along with its edges, color the rest of G with 5 colors (surely possible by the inductive hypothesis), then replace x. Since x is adjacent to only 4 other vertices, there is a color with which x can be colored. [picture]

All that's left is the case where x has degree 5.

Proof of 5CT, cont.

If x has degree 5 we do the same thing: Remove x from G, color $G–x$ with 5 colors, then put x back. If the five neighbors of x have only four colors, there's a color left for x and we're done.

The problematic case is when the five neighbors of x, call them [in order!] v_1, v_2, v_3, v_4, v_5, have five different colors in the 5-coloring of $G–x$, so that there's no color left for x. Let c_i be the color of v_i for each i. [picture]

Let H be the subgraph of $G–x$ consisting of all vertices colored c_1 or c_3 plus all edges between these vertices. Suppose first that v_1 and v_3 are in *different* connected components of H [picture]. Then we recolor all vertices in v_1's component by swapping c_1 and c_3 everywhere. This new coloring is a legal coloring of $G–x$.
[Proof: The only possible problem is if there is now an edge joining two nodes with the same color, c_1 or c_3. This can happen only in v_1's component in H since the rest of H is unchanged. But c_1 and c_3 are swapped everywhere in that component, so it doesn't happen.]
Now v_1 has color c_3 so we can color x with c_1 and we're done.

Proof of 5CT, cont.

The last case to consider is that v_1 and v_3 are in the *same* connected component of H. This means that there is a path P connecting them. Since the vertices of P lie entirely within H, they are all colored c_1 or c_3. But then the colors c_1 and c_3 must *alternate* along P.
Such a path P is called a *Kempe chain*. [picture]

[Digression: Kempe was the first person to "prove" the 4CT, in 1879, but his proof was shown to be flawed in 1890. The ideas from his proof are those used here to prove 5CT, and they're also used in the proof of 4CT.]

If we add x to P we get a cycle. This cycle must enclose either v_2 or both v_4 and v_5. [picture] In either case, there cannot be a Kempe chain from v_2 to v_4 of vertices colored alternately c_2 and c_4—any such chain would have to intersect P! [Why is this impossible?]

So v_2 and v_4 must lie in different connected components of the subgraph of G–x made up of the vertices colored c_2 or c_4. We can therefore swap c_2 and c_4 in one of these components as before, freeing up a color for x. QED

How to prove 4CT

A *configuration* of a planar graph, loosely, consists of some vertices, the edges between those vertices, and the edges connecting these vertices to the rest of the graph (but not the vertices on the other sides of those edges, which is why a configuration is not a subgraph).
[blackboard picture of some configurations]

A *set* of configurations is called *unavoidable* if every planar graph must contain at least one member of the set.
[blackboard examples]

We call a configuration C *reducible* if it satisfies the following property: Suppose G is any planar graph that contains C and is not 4-colorable. Then there is a graph with fewer vertices than G that is also not 4-colorable. That is, a reducible configuration can be used to *reduce* the number of vertices of a non-4-colorable graph.

Example: A configuration consisting of a single, isolated vertex is obviously reducible; so is an isolated copy of K_4. [blackboard: other examples]

Proof of the 4CT

To prove the Four Color Theorem, Appel and Haken—with the help of a computer and a graduate student—constructed an *unavoidable set of configurations, each of which is reducible*. Using this set (call it *S*) we finish off the proof like this:

Suppose the Four Color Theorem is false. Let *G* be a graph that is not 4-colorable and has the fewest possible number of vertices. (That is, any graph with fewer vertices than *G* must be 4-colorable. Clearly G exists.)

Since the set of configurations *S* is unavoidable, *G* must contain a configuration from *S*, call it *C*. But since *C* is reducible, we can construct a graph with fewer vertices than *G* that is not four-colorable! Contradiction.

The unavoidable set of reducible configurations constructed by Appel and Haken had nearly 2000 members, later reduced to about 1500. A newer proof (Robertson, Sanders, Seymour, Thomas) uses only 633 reducible configurations.